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A NOTE ON GENERALIZED $*$-DERIVATIONS OF PRIME $*$-RINGS

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Abstract. The aim of the present paper is to establish some results involving generalized $*$-derivations in $*$-rings and investigate the commutativity of prime $*$-rings admitting generalized $*$-derivations of R satisfying certain identities and some related results have also been discussed.

1. Introduction

Over the last few decades, several authors have investigated the relationship between the commutativity of the ring R and certain specific types of derivations of R. The first result in this direction is due to E. C. Posner [8] who proved that if a ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. This result was subsequently, refined and extended by a number of authors. In [7], Bresar and Vuckman showed that a prime ring must be commutative if it admits a nonzero left derivation. Recently, many authors have obtained commutativity theorems for prime and semiprime rings admitting derivation, generalized derivation. Bresar and Vukman [5] studied the notions of a $*$-derivation and a Jordan $*$-derivation of R. The aim of the present paper is to establish some results involving generalized $*$-derivations in $*$-rings and investigate the commutativity of prime $*$-rings admitting generalized $*$-derivations of R satisfying certain identities and some related results have also been discussed.

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2. Preliminaries

Throughout R will represent an associative ring with center $Z(R)$. For all $x, y \in R$, as a usual commutator, we shall write $[x, y] = xy - yx$, and $x \circ y = xy + yx$. Also, we make use of the following two basic identities without any specific mention:

$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$

$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$$

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z.$$

Let R is a ring. Then R is *prime* if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. An additive mapping $d : R \to R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $x \to x^*$ of R into itself is called an *involution* if the following conditions are satisfied (i) $(xy)^* = y^*x^*$, (ii) $(x^*)^* = x$ for all $x, y \in R$. A ring equipped with an involution is called an *$*$-ring* or *ring with involution*. Let R be a $*$-ring. An additive mapping $d : R \to R$ is called an *$*$-derivation* if $d(xy) = d(x)y^* + xd(y)$ holds for all $x, y \in R$. An additive mapping $d : R \to R$ is called a *reverse $*$-derivation* if $d(xy) = d(y)x^* + yd(x)$ holds for all $x, y \in R$. An additive mapping $F : R \to R$ is called a *generalized derivation* if there exists a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Let R be an $*$-ring. An additive mapping $F : R \to R$ is called a *generalized $*$-derivation* if there exists an $*$-derivation d such that $F(xy) = F(x)y^* + xd(y)$ for all $x, y \in R$. An additive mapping $F : R \to R$ is called a *generalized reverse $*$-derivation* if there exists an $*$-derivation d such that $F(xy) = F(y)x^* + yd(x)$ for all $x, y \in R$. An additive mapping $F : R \to R$ is called a *right $*$-multiplier* of R if $F(xy) = x^*F(y)$ for all $x, y \in R$. Also, an additive mapping $F : R \to R$ is called a *left $*$-multiplier* of R if $F(xy) = F(x)y^*$ for all $x, y \in R$. An additive mapping $F : R \to R$ is called a *reverse left $*$-multiplier* of R if $F(xy) = F(y)x^*$ for all $x, y \in R$ and an additive mapping $F : R \to R$ is called a *reverse right $*$-multiplier* of R if $F(xy) = y^*F(x)$ for all $x, y \in R$.

3. Generalized $*$-derivations of prime $*$-rings

LEMMA 3.1. *Let R be a semiprime $*$-ring and $a \in R$. If R admits a generalized $*$-derivation F associated with an $*$-derivation d of R and $F(x) = [x, a]$ for all $x \in R$, then $F = 0$.*

Proof. By hypothesis, we have

$$F(xy) = F(x)y^* + xd(y),\ \forall\ x, y \in R. \tag{3.1}$$

Replacing y by yz in (3.1), we have

$$F(x(yz)) = F(x)z^*y^* + xd(y)z^* + xyd(z),\ \forall\ x, y, z \in R \tag{3.2}$$

and

$$F((xy)z) = F(x)y^*z^* + xd(y)z^* + xyd(z),\ \forall\ x, y, z \in R. \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$F(x)[y^*, z^*] = 0,\ \forall\ x, y, z \in R. \tag{3.4}$$

Substituting y^* for y and z^* for z in (3.4), we have $F(x)[y, z] = 0$ for all $x, y, z \in R$. Again, replacing y by yx in the last relation, we have $F(x)y[x, z] = 0$ for all $x, y, z \in R$. This implies that $F(x)R[x, z] = \{0\}$ for all $x, z \in R$. Taking a in stead of z in this relation, we get $F(x)R[x, a] = \{0\}$ for all $x \in R$. If $F(x) = [x, a]$, then, we have $[a, x]R[a, x] = \{0\}$ for all $x \in R$. Since R is semiprime, we have $[x, a] = 0$, that is, $F(x) = [x, a] = 0$ for all $x \in R$. This implies that $F = 0$. $\square$

THEOREM 3.2. *Let R be a semiprime $*$-ring. If R admits a generalized $*$-derivation F associated with $*$-derivation d of R, then F maps from R to $Z(R)$.*

Proof. By hypothesis, we have

$$F(xy) = F(x)y^* + xd(y),\ \forall\ x, y \in R. \tag{3.5}$$

Replacing y by yz in (3.5), we have

$$F(x(yz)) = F(x)z^*y^* + xd(y)z^* + xyd(z),\ \forall\ x, y, z \in R. \tag{3.6}$$

On the other hand,

$$F(xyz) = F((xy)z) = F(x)y^*z^* + xd(y)z^* + xyd(z),\ \forall\ x, y, z \in R. \tag{3.7}$$

Combining (3.6) with (3.7), we have $F(x)[y^*, z^*] = 0$ for all $x, y, z \in R$. Substituting y^* for y and z^* for z in this relation, we have $F(x)[y, z] = 0$ for all $x, y, z \in R$. Taking $yF(x)$ instead of y in the last relation, we have

$$F(x)y[F(x), z] = 0,\ \forall\ x, y \in R. \tag{3.8}$$

Multiplying the left side of (3.8) by $zF(x)$, we have

$$zF(x)F(x)y[F(x), z] = 0,\ \forall\ x, y, z \in R. \tag{3.9}$$

Again, multiplying the left side of (3.8) by $F(x)z$, we have

$$F(x)zF(x)y[F(x), z] = 0,\ \forall\ x, y, z \in R. \tag{3.10}$$

Subtracting (3.9) from (3.10), we have$[F(x), z]F(x)y[F(x), z] = 0$ for all $x, y, z \in R$. Hence we have $[F(x), z]R[F(x), z] = \{0\}$ for all $x, z \in R$. Since R is semiprime, we have $[F(x), z] = 0$ for all $x, z \in R$. Therefore F is a mapping form R into $Z(R)$. □

THEOREM 3.3. *Let R be a prime $*$-ring. If R admits a generalized $*$-derivation F associated with an $*$-derivation d such that $F(x) \neq x$ and $F(xy) = F(x)F(y)$ for all $x, y \in R$, then $d = 0$.*

Proof. By hypothesis, we have

$$F(xy) = F(x)y^* + xd(y) = F(x)F(y),\ \forall\ x, y \in R. \tag{3.11}$$

Replacing x by xz in (3.11), we have

$$\begin{aligned} F(x)F(z)y^* + xzd(y) &= F(x)F(z)F(y) \\ &= F(x)F(zy) = F(x)(F(z)y^* + zd(y)), \end{aligned}$$

which implies that $(x - F(x))zd(y) = 0$ for all $x, y, z \in R$. Hence we have $(x - F(x))Rd(y) = \{0\}$ for all $x, y \in R$. Since R is prime, we have $x - F(x) = 0$ or $d(y) = 0$ for all $x, y \in R$. But $F(x) \neq x$, and so $d(y) = 0$ for all $y \in R$, that is, $d = 0$. □

THEOREM 3.4. *Let R be a prime $*$-ring. If R admits a generalized $*$-derivation F associated with an $*$-derivation d of R and $F(xy) = F(y)F(x)$ for all $x, y \in R$, then $d = 0$.*

Proof. By hypothesis, we have

$$F(xy) = F(x)y^* + xd(y) = F(y)F(x),\ \forall\ x, y \in R. \tag{3.12}$$

Replacing x by xy in (3.12), we have

$$F(xy)y^* + xyd(y) = F(y)F(xy) = F(y)(F(x)y^* + xd(y)),$$

which implies that $F(y)F(x)y^* + xyd(y) = F(y)F(x)y^* + F(y)xd(y)$ for all $x, y \in R$. Hence we have

$$xyd(y) = F(y)xd(y),\ \forall\ x, y \in R. \tag{3.13}$$

Taking wx instead of x in (3.13) and using (3.13), we have $wF(y)xd(y) = F(y)wxd(y)$ for all $x, y, w \in R$. This implies that $[w, F(y)]xd(y) = 0$, and so $[w, F(y)]Rd(y) = \{0\}$ for all $w, y \in R$. Since R is prime, we have $d(y) = 0$ or $[w, F(y)] = 0$ for all $y, w \in R$. Let $K = \{y \in R | d(y) = 0\}$ and $L = \{y \in R | [w, F(y)] = 0, \forall\ w, y \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d = 0$. If $L = R$, then $[w, F(y)] = 0$ for all $w, y \in R$. Hence we have $F(y) \in Z(R)$, and so $F(xy) = F(y)F(x) = F(x)F(y)$ for all $x, y \in R$. Since F acts an endomorphism of R, it follows that $d = 0$ via Theorem 3.3. $\square$

THEOREM 3.5. *Let R be a noncommutative prime $*$-ring. If R admits a generalized $*$-derivation F associated with an $*$-derivation d and $F(x) \in Z(R)$ for all $x, y \in R$, then $d = 0$.*

Proof. By hypothesis, we have

$$[F(xy), z] = 0, \ \forall\, x, y, z \in R, \tag{3.14}$$

which implies that $[F(x)y^* + xd(y), z] = 0$, and so

$$F(x)[y^*, z] + x[d(y), z] + [x, z]d(y) = 0$$

for all $x, y, z \in R$. Replacing z by y^* in the above relation, we have $x[d(y), y^*] + [x, y^*]d(y) = 0$ for all $x, y \in R$, that is,

$$xd(y)y^* = y^*xd(y), \ \forall\, x, y \in R. \tag{3.15}$$

Substituting xz for x in (3.15), we have $xzd(y)y^* = y^*xzd(y)$ for all $x, y, z \in R$. Using the relation (15), we have $xy^*zd(y) = y^*xzd(y)$ for all $x, y, z \in R$, that is, $[x, y^*]zd(y) = 0$ for all $x, y, z \in R$. Hence $[x, y^*]Rd(y) = \{0\}$ for all $x, y \in R$. Since R is prime, we have $[x, y^*] = 0$ or $d(y) = 0$ for all $x, y \in R$. But R is noncommutative, and so $d(y) = 0$ for all $y \in R$, which means that $d = 0$. $\square$

THEOREM 3.6. *Let R be a semiprime $*$-ring. If R admits a generalized reverse $*$-derivation F associate with a nonzero reverse $*$-derivation d, then $[d(x), z] = 0$ for all $x, z \in R$.*

Proof. By hypothesis, we have

$$F(xy) = F(y)x^* + yd(x), \ \forall\, x, y \in R. \tag{3.16}$$

Replacing x by xz in (3.16), we have

$$\begin{aligned} F((xz)y) &= F(y)(xz)^* + yd(xz) \\ &= F(y)z^*x^* + y(d(z)x^* + zd(x)) \\ &= F(y)z^*x^* + yd(z)x^* + yzd(x) \end{aligned} \tag{3.17}$$

for every $x, y, z \in R$. On the other hand, we have

$$\begin{aligned} F(x(zy)) &= F(zy)x^* + zyd(x) \\ &= (F(y)z^* + yd(z))x^* + zyd(x) \\ &= F(y)z^*x^* + yd(z)x^* + zyd(x) \end{aligned} \tag{3.18}$$

for every $x, y, z \in R$. Comparing (3.17) and (3.18), we get $[y, z]d(x) = 0$ for all $x, y, z \in R$. Substituting $d(x)y$ for y in this relation, we obtain

$$[d(x), z]yd(x) = 0, \ \forall\ x, y, z \in R. \tag{3.19}$$

Multiplying the right side of (3.19) by $zd(x)$, we have

$$[d(x), z]yd(x)zd(x) = 0, \ \forall\ x, y, z \in R. \tag{3.20}$$

Multiplying the right side of (3.19) by $d(x)z$, we have

$$[d(x), z]yd(x)d(x)z = 0, \ \forall\ x, y, z \in R. \tag{3.21}$$

Subtracting (3.20) from (3.21), we have $[d(x), z]yd(x)[d(x), z] = 0$ for all $x, y, z \in R$. This implies that $[d(x), z]R[d(x), z] = \{0\}$ for all $x, z \in R$. Since R is semiprime, we have $[d(x), z] = 0$ for all $x, z \in R$. □

THEOREM 3.7. *Let R be a noncommutative prime $*$-ring. If R admits a generalized reverse $*$-derivation F associated with a nonzero reverse $*$-derivation d of R, then F is a reverse left $*$-multiplier of R.*

Proof. By Theorem 3.6, we have

$$[y, z]d(x) = 0, \ \forall\ x, y, z \in R. \tag{3.22}$$

Replacing y by xy in (3.22), we have $[xy, z]d(x) = 0$, and so $[x, z]yd(x) = 0$ for all $x, y, z \in R$. This implies that $[x, z]Rd(x) = 0$ for all $x, z \in R$. Since R is prime, we have $[x, z] = 0$ or $d(x) = 0$ for all $x, z \in R$. Let $K = \{x \in R \mid d(x) = 0\}$ and $L = \{x \in R \mid [x, z] = 0, \forall\ z \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $L = R$ or $K = R$. In the former case, R is commutative, contradiction. On the other hand, if $K = R$, then $d(x) = 0$ for all $x \in R$, that is, $d = 0$. Hence $F(xy) = F(y)x^*$ for all $x, y \in R$. This implies that F is a reverse left $*$-multiplier of R. □

THEOREM 3.8. *Let R be a prime $*$-ring. If R admits a generalized $*$-derivation F associated with an $*$-derivation d such that $F([x, y]) = 0$ for all $x, y \in R$, then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$F([x, y]) = 0, \ \forall\ x, y \in R. \tag{3.23}$$

Replacing x by xy in (3.23), we have

$$F([x, y]y) = F([x, y])y^* + [x, y]d(y) = 0$$

for all $x, y \in R$. By the relation (3.23), we have $[x, y]d(y) = 0$ for all $x, y \in R$. Substituting sx for x in this relation, we have $[s, y]xd(y)$ for all $x, y, s \in R$. This implies that $[s, y]Rd(y) = \{0\}$ for all $s, y \in R$. Since R is prime, we have $[s, y] = 0$ or $d(y) = 0$ for all $s, y \in R$. Let $K = \{y \in R \mid d(y) = 0\}$ and $L = \{y \in R \mid [s, y] = 0, \forall\ s \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d = 0$, and in second case, R is commutative. □

THEOREM 3.9. *Let R be a prime $*$-ring. If R admits a generalized $*$-derivation F associated with an $*$-derivation d such that $F(x \circ y) = 0$ for all $x, y \in R$, then $d = 0$ or R is commutative.*

Proof. By hypothesis, we have

$$F(x \circ y) = 0,\ \forall\ x, y \in R. \tag{3.24}$$

Replacing x by xy in (3.24), we have

$$F((x \circ y)y) = F(x \circ y)y^* + (x \circ y)d(y) = 0$$

for all $x, y \in R$. By the relation (3.24), we have $(x \circ y)d(y) = 0$ for all $x, y \in R$. Substituting sy for x in this relation, we have $(s \circ y)yd(y)$ for all $y, s \in R$. This implies that $(s \circ y)Rd(y) = \{0\}$ for all $s, y \in R$. Since R is prime, we have $(s \circ y) = 0$ or $d(y) = 0$ for all $s, y \in R$. Let $K = \{y \in R \mid d(y) = 0\}$ and $L = \{y \in R \mid s \circ y = 0, \forall\ s \in R\}$. Then K and L are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d = 0$. On the other hand, if $L = R$, then we have $s \circ y = 0$ for all $s, y \in R$. Replacing s by sz in the last relation and using the fact that $ys = -sy$, we obtain $s[z, y] = 0$ for all $s, y, z \in R$. That is, $R[z, y] = \{0\}$. This implies that $xR[z, y] = \{0\}$ for $0 \neq x \in R$. Since R is prime, we have $[z, y] = 0$ for all $y, z \in R$, which means that R is commutative. □

THEOREM 3.10. *Let R be a 2-torsion free prime $*$-ring. If R admits a generalized $*$-derivation F associated with an $*$-derivation d such that $F(x \circ y) = [x, y]$ for all $x, y \in R$, then $d = 0$.*

Proof. By hypothesis, we have

$$F(x \circ y) = [x, y],\ \forall\ x, y \in R. \tag{3.25}$$

Replacing y by yx in (3.25), we have

$$F((x \circ y)x) = F(x \circ y)x^* + (x \circ y)d(x) = [x, y]x$$

for all $x, y \in R$. By the relation (3.25), we get

$$[x, y]x^* + (x \circ y)d(x) = [x, y]x,\ \forall\ x, y \in R. \tag{3.26}$$

Substituting y for x in this relation, we have $(y \circ y)d(y) = 0$ for all $x, y \in R$. This implies that $2y^2 d(y) = \{0\}$ for all $y \in R$. Since R is 2-torsion free, we have $y^2 d(y) = 0$ for all $y \in R$. This implies that $yRd(y) = \{0\}$ for all $y \in R$. Since R is prime, we obtain $y = 0$ or $d(y) = 0$ for all $y \in R$. In the former case, $y = 0$ for all $y \in R$, a contradiction. Hence $d(y) = 0$ for all $y \in R$, that is, $d = 0$. □

Theorem 3.11. *Let R be a 2-torsion free prime $*$-ring. If R admits a generalized $*$-derivation F associated with an $*$-derivation d such that $F(x \circ y) = -[x, y]$ for all $x, y \in R$, then $d = 0$.*

Proof. Using the similar technique with necessary variations in the above theorem, we get the required result. □

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